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# A NEW APPROACH FOR OBTAINING NORMAL FORMS OF NON-LINEAR SYSTEMS

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In this paper, a modified approach for obtaining normal forms of non-linear dynamical systems is described. This approach provides a number of significant advantages over the existing normal form theory, and improves the associated calculations. A brief discussion concerning the application of the new approach to high-dimensional systems is also presented. To illustrate the new approach, three examples are presented.



## 1. INTRODUCTION

Researchers in various fields of science and engineering are becoming increasingly interested in non-linear dynamical systems which are liable to exhibit very complex motions. Thus, bifurcations and chaos are under intense study in a number of contexts. Studies concerning non-linear behavior can be classified into two major categories: local and global behavior. For example, saddle-node, Hopf bifurcation and other post-critical behavior of non-linear systems can be analyzed locally in the vicinity of a critical point. On the other hand, homoclinic and heteroclinic orbits and chaos are essentially global and require different approaches.

Local studies usually involve an appropriate simplification of the governing non-linear equations in the vicinity of a point of interest, so as to facilitate the analysis. Various methods can serve the simplification purpose, such as the averaging method [1, 2, 9, 12], normal forms [2, 3, 10], the Lyapunov–Schmidt method [4, 5], the method of succession functions [6] and the intrinsic harmonic balancing technique [7]. Most of these methods are not easy to use and require a great deal of work. For example, normal form theory aims at simplifying the governing equations of the system and thus simplifying the analysis; but generating a complete normal form is not that simple. As discussed in reference [8], many researchers, such as V. I. Arnold [3] and F. Takens [11], have contributed to the development of normal forms. A detailed discussion and summary about this can be found in reference [10]. Some researchers have tried to employ the averaging methods to obtain the normal forms [9], but the calculation of coefficients of normal forms in reference [9] appears to be even harder than the current normal form theory. It is noted that the basic terms of a normal form is not difficult to determine. Normal forms are composed of resonant monomials which are easy to obtain for a given non-linear system. However,

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obtaining the coefficients associated with each term of the normal forms may be quite cumbersome. In spite of the efforts, the existing methods are not very convenient to apply, requiring a great deal of labor. Some modern but abstract mathematical theories, such as representation theory [10], have to be employed to obtain the coefficients of normal forms. However, representation theory requires the solution of a matrix equation of size $nC^{n-1}_{n+k-1} \times nC^{n-1}_{n+k-1}$ (where $C^{n-1}_{n+k-1} = (n+k-1)!/(n-1)!k!$), which becomes increasingly difficult as $k$ increases [10]. Recent developments on symbolic computations are expected to facilitate both the determination of basic terms as well as the associated coefficients. However, the existing methodology concerning normal forms does not always lend itself to such symbolic calculations [13]. It is observed that the full calculation of the normal forms of order 6 for a particular example, using MACSYMA, "strained the memory capacity of a modern minicomputer" [15]. It seems that the methodology itself has to be modified if symbolic computations are to be utilized efficiently. Indeed, this is one of the underlying motivations for developing a modified approach to determine normal forms, which is introduced in this paper for the first time.

In comparison to the existing normal form theory and the approaches by averaging methods [9], the new approach presented here has some notable differences and advantages: (1) it is simple to apply to specific problems—the calculations of both basic terms of normal forms and the associated coefficients are easy, and symbolic calculations become possible; (2) it is conveniently applicable to higher-dimensional systems as well—here, it is recognized that applying the existing methodology to systems with higher than two dimensions can be extremely involved; (3) the new approach is also applicable to non-autonomous non-linear systems. It is important to note that the results of the new approach have been shown to be equivalent to those obtained by the existing normal form theory.

In this introductory paper, only some aspects of the approach are discussed. A more complete discussion will be given in subsequent papers.

Three examples are presented to illustrate some of the advantages of the new approach. Only two dimensional examples are analyzed here. The results clearly show the equivalence of the new approach to other normal form methods. In the first example, the coefficients of normal forms of order 3 for a general quadratic and cubic system are obtained in 1 s using MAPLE with the aid of a PC (CPU 200) computer. In the second example, the coefficients of normal forms of order 7 in a multiple parameter system are obtained in 2 s. In the third example, the coefficients of normal forms of order 5 in the Duffing equation are obtained in 1 s. The authors are not aware of any existing method that is capable of yielding the coefficients of normal form as conveniently and as fast as the approach presented in this paper.

## 2. BASIC CONCEPTS OF NORMAL FORM THEORY

First, some basic concepts, which will be used in this paper, are presented. Consider the following equation

$$\dot{x} = f(x) = Ax + f^2(x) + \cdots + f^r(x) + O(|x|^{r+1}), \tag{1}$$

where $A$ is a matrix, $f^k \in H^k_n$, and $H^k_n$ is a vector space of homogeneous polynomials of order $k$ in $n$ variables, with values in $C^n$, $k = 2, 3, \ldots, r$.

Suppose that $A$ can be transformed into diagonal form. Consider a series of near identity transformations

$$x = y + h^k(y), \qquad k \geqslant 2, \tag{2}$$

where $h^k \in H_n^k$ are undefined functions that will be determined such that the terms of order $k$ in the transformed form will be simplified as resonant polynomial of order $k$. Substituting equation (2) into equation (1) results in

$$\frac{\mathrm{d}}{\mathrm{d}t}[y + h^k(y)] = A[y + h^k(y)] + f^2(y + h^k(y)) + \cdots + f^k(y + h^k(y)) + O(|y|^{k+1}) \quad (3)$$

and using Taylor expansion (about point $y$), gives

$$\dot{y} = Ay + f^2(y) + \cdots + f^{k-1}(y) + (f^k(y) - ad_A^k h^k(y)) + O(|y|^{k+1}), \quad (4)$$

where $ad_A^k$ is the linear operator $ad_A^k$: $H_n^k \to H_n^k$, defined by

$$(ad_A^k h^k)(y) = Dh^k(y)Ay - Ah^k(y) \quad (5)$$

and $Dh^k(y)$ is the Jacobian matrix of $h^k(y)$.

Equation (4) indicates that the terms with the order less than $k$ do not change in form: only those terms with the order equal to or more than $k$ change in their forms. This is the simplest form for a polynomial of order $k$ if

$$f^k(y) - ad_A^k h^k(y) = 0, \qquad k \geqslant 2. \quad (6)$$

Let us denote $M^k$ as the range of the operator $ad_A^k$ in $H_n^k$ and let $N^k$ be any complementary subspace to $M^k$ in $H_n^k$. Then,

$$H_n^k = M_n^k \oplus N_n^k, \qquad k \geqslant 2. \quad (7)$$

If $f^k(y) \in M^k$, there exists $h^k(y) \in H_n^k$ such that

$$ad_A^k h^k(y) = f^k(y), \qquad k \geqslant 2. \quad (8)$$

This means that the polynomial of order $k$ in equation (4) can be transformed to zero. Otherwise, we can only find $h^k \in H_n^k$, which leads to $f^k(y) - ad_A^k h^k(y) \in N^k$. Suppose that $f^k(y) = \xi^k(y) + g^k(y)$, where $\xi^k(y) \in M^k$, $g^k(y) \in N^k$. If we choose $h^k$, which leads to $ad_A^k(h^k(y)) = \xi^k(y)$, then equation (4) can be transformed into the form

$$\dot{y} = Ay + f^2(y) + \cdots + f^{k-1}(y) + g^k(y) + O(|y|^{k+1}). \quad (9)$$

One can now state the following theorem [2, 10].

*Theorem* 2.1. Let $f$: $C^n \to C^n$ be a $C^{r+1}$ vector field with $f(0) = 0$ and $Df(0) = A$. Let the decomposition (7) of $H_n^k$ be given for $k = 2, \ldots, r$. Then there exists a sequence of near identity transformations $x = y + h^k(y)$, $y \in \Omega_k$, where $h^k \in H_n^k$ and $\Omega_k$ is a neighborhood of the origin, $\Omega_{k+1} \subseteq \Omega_k$, $k = 2, \ldots, r$, such that equation (1) is transformed into

$$\dot{y} = Ay + g^2(y) + \cdots + g^k(y) + O(|y|^{k+1}), \quad (10)$$

where $g^k(y) \in N^k$ for $k = 2, 3, \ldots, r$.

The following truncated equation of equation (10),

$$\dot{y} = Ay + g^2(y) + \cdots + g^k(y), \quad (11)$$

is called a normal form of equation (1).

The normal form is not unique for a fixed matrix $A$. In fact, it depends on the choices of complementary subspace $N^k$. Usually, a standard basis is chosen the basis element of which is $e_j = (0, \ldots, 1, \ldots, 0)^{\mathrm{T}}$, in which only the $j$th component is 1 and all other components are zero.

Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of $A$. Then the following relations are called resonant conditions:

$$\lambda_s - \sum_{i=1}^{n} m_i \lambda_i = 0, \tag{12}$$

where

$$\bar{m} = \sum_{i=1}^{n} m_i \geqslant 2.$$

Let $(x_1, \ldots, x_n)$ be co-ordinates with respect to the standard basis $\{e_1, \ldots, e_n\}$ of $C^n$ in which the matrix $A$ is in diagonal form the elements of which are $(\lambda_1, \ldots, \lambda_n)$. Then a monomial $x^m e_s$ ($\bar{m} = k \geqslant 2$ and $1 \leqslant s \leqslant n$) is called a resonant monomial of order $k$ if and only if equation (12) holds for $m_i$ and $s$, and one can state the following result [10].

*Theorem* 2.2. $x^m e_s \in N^k$, if and only if

$$\lambda_{ms} = (m, \lambda) - \lambda_s = 0, \tag{13}$$

where

$$(m, \lambda) = \sum_{i=1}^{n} m_i \lambda_i.$$

## 3. A NEW APPROACH FOR OBTAINING NORMAL FORMS

In this section, some new ideas will be introduced into existing normal form theory, which make the calculation of normal forms relatively simple. First, the procedures of the existing normal form theory are discussed briefly.

Consider the equation

$$\dot{z} = Az + F^2(z) + F^3(z) + \text{h.o.t.}, \tag{14}$$

where $z \in C^2$; $F^k \in H_2^k$, $k = 2, 3, \ldots$; "h.o.t." means higher order terms; and $H_2^k$ is the bi-variate polynomial space of order $k$. They are defined by

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \qquad A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad F^k(z) = \begin{bmatrix} \sum\limits_{i+j=k} a_{ij} z_1^i z_2^j \\ \\ \sum\limits_{i+j=k} b_{ij} z_1^i z_2^j \end{bmatrix}, \qquad i, j = 0, \ldots, k, \quad k = 2, 3,$$

where $a_{ij}$ and $b_{ij}$ are constants and $\lambda_1$ and $\lambda_2$ are the eigenvalues.

Suppose that

$$z = y + P^2(y), \qquad P^2 \in H_2^2, \tag{15}$$

where $P^2(y)$ is an undefined function, which will be determined such that the terms of order 2 in the transformed form will be simplified as resonant polynomial of order 2.

Substituting equation (15) into equation (14) results in

$$\dot{y} = Ay + F_1^2(y) + F_1^3(y) + \text{h.o.t.}, \tag{16}$$

where $F_1^2 = F^2 + AP^2 - DP^2Ay$ and $F_1^3 = F^3 + DF^2P^2 - DP^2F_1^2$.

Suppose that $F_1^2(y) = G^2(y)$ in equation (16), where $G^2(y)$ is the resonant polynomial of order 2. Solving this equation for $P^2(y)$, then, the coefficients in $G^2(y)$ can be obtained. Substituting $P^2(y)$ into $F_1^3(y)$ defines $F_1^3$ as $F_1^3(y)$. Then, suppose that

$$y = x + P^3(x), \qquad P^3 \in H_2^3, \tag{17}$$

where $P^3(y)$ is an undefined function, which will be determined such that the terms of order 3 in the transformed form will be simplified as resonant polynomial of order 3.

Substituting equation (17) into equation (16) results in

$$\dot{x} = Ax + G^2(x) + F_2^3(x) + \text{h.o.t.} \tag{18}$$

where $F_2^3(x) = F_1^3(x) + AP^3 - DP^3Ax$.

Suppose that $F_2^3(x) = G^3(x)$, where $G^3(x)$ is the resonant polynomial of order 3. Solving for $P^3(x)$, then, the coefficients in $G^3(x)$ can be obtained. Thus the normal form of equation (14) is given by

$$\dot{x} = Ax + G^2(x) + G^3(x) + \text{h.o.t.}, \tag{19}$$

Transforming the above equation into polar co-ordinates, one has

$$\dot{r} = a_1r^2 + a_2r^3 + O(r^4), \qquad \dot{\theta} = \omega + b_1r + b_2r^2 + O(r^3), \tag{20}$$

where $a_1$, $a_2$, $b_1$ and $b_2$ are related to the coefficients in functions $F^2$ and $F^3$.

It is generally agreed that it is not difficult to obtain the *form* of the normal form of equation (14), but it is not easy to determine the coefficients of the normal form; i.e., the coefficients of the resonant monomials. Some researchers have employed modern mathematical theories to obtain the functions $P^k(x)$, and this needs a lot of work. For example, in order to obtain the function $P^k(x)$, a lot of matrix calculations, with the size of matrix $nC^{n-1}_{n+k-1} \times nC^{n-1}_{n+k-1}$, where $n$ is the dimension, are required. It is stated in reference [10] that, "as $k$ increases, the calculations become generally more and more difficult". To overcome this disadvantage, a new approach is presented.

Next, the new approach for obtaining normal forms is introduced. Consider a series of near identity transformations, which are similar to equations (15) and (17) in $C^2$, given by

$$z = y + P^k(y), \qquad k = 2, 3, \ldots,$$

where $P^k(y)$ are undefined functions, which will be determined such that the terms of order $k$ in the transformed form will be simplified as a resonant polynomial of order $k$.

Substituting the above transformations into equation (14) results in

$$\dot{y} = Ay + F_1^2(y) + F_1^3(y) + \text{h.o.t.} \qquad \text{when } k = 2,$$

$$\dot{y} = Ay + F_1^2(y) + F_2^3(y) + \text{h.o.t.} \qquad \text{when } k = 2 \text{ and } k = 3. \tag{21}$$

Suppose that $F_{k-1}^k(y) = G^k(y)$ in equation (21), where $G^k(y)$ are the resonant polynomials of order $k$. Then, solving the above equation for $P^k(y)$, the coefficients in $P^k(y)$ and $G^k(y)$ can be obtained by the following steps.

Introducing the transformation

$$y = e^{At}z \tag{22}$$

into equations (21), one has

$$\dot{z} = e^{-At}[F_1^2(e^{At}z) + F_2^3(e^{At}z)] + \text{h.o.t.}, \tag{23}$$

where

$$e^{At}z = \begin{pmatrix} e^{\lambda_1 t}z_1 \\ e^{\lambda_2 t}z_2 \end{pmatrix}.$$

Suppose that

$$F^k_{k-1}(y) = \begin{pmatrix} F^k_{k-1(1)}(y) \\ F^k_{k-1(2)}(y) \end{pmatrix} = \begin{bmatrix} \sum_{m+n=k} a^{k-1}_{mn(1)} y_1^m y_2^n \\ \\ \sum_{m+n=k} a^{k-1}_{mn(2)} y_1^m y_2^m \end{bmatrix}.$$

Then $e^{-At}F^k_{k-1(q)}(e^{At}z)$ can be expressed as

$$e^{-At}F^k_{k-1(q)}(e^{At}z) = \sum_{m+n=k} e^{-\lambda_q t} a^{k-1}_{mn(q)}\, e^{m\lambda_1 t} z_1^m\, e^{n\lambda_2 t} z_2^n = \sum_{m+n=k} a^{k-1}_{mn(q)}\, e^{(m\lambda_1 + n\lambda_2 - \lambda_q)t} z_1^m z_2^n, \qquad (24)$$

where $\lambda_q = \lambda_1, \lambda_2$; $q = 1, 2$.

According to the assumption $F^k_{k-1}(y) = G^k(y)$, functions $F^k_{k-1}(y)$ are composed of resonant monomials, in which

$$m\lambda_1 + n\lambda_2 = \lambda_q. \qquad (25)$$

According to equations (23)–(25), one has

$$e^{-At}F^k_{k-1}(e^{At}z) = F^k_{k-1}(z) = \underset{t}{M}\{e^{-At}F^k_{k-1}(e^{At}z)\} \qquad (26)$$

and

$$\begin{aligned} \dot{z} &= \underset{t}{M}\{e^{-At}F^2_1(e^{At}z)\} + \underset{t}{M}\{e^{-At}F^3_2(e^{At}z)\} + \text{h.o.t.} \\ &= F^2_1(z) + F^3_2(z) + \text{h.o.t.}, \end{aligned}$$

where $\underset{t}{M}\{f(z, t)\} = (1/T)\int_0^T f(z, t)\,dt$ denotes explicit time averaging of function $f(z, t)$; $T$ is the period. Similarly, one has

$$e^{At}F^k_{k-1}(e^{-At}z) = F^k_{k-1}(z). \qquad (27)$$

Thus, equation (23) can be expressed as

$$\dot{z} = G^2(z) + G^3(z) + \text{h.o.t.} \qquad (28)$$

Carrying out the inverse transformation $z = e^{-At}x$ in equation (28), according to equation (27), one has

$$\dot{x} = Ax + G^2(x) + G^3(x) + \text{h.o.t.} \qquad (29)$$

This is the normal form of equation (14).

From the procedures leading to $F^k_m(y)$ and $P^k(y)$ ($m \leqslant k-1$), one can see that functions $F^k_m(y)$ and $P^k(y)$ obtained from existing normal form theory and those from the new approach are identical to each other. Then, one can obtain the following conclusion, evidently: *The results of existing normal form theory are identical to those of the new approach*.

In order to obtain the normal forms and the related coefficients more conveniently, consider the relations

$$e^{-At}(DP^k(e^{At}z)A\,e^{At}z - AP^k(e^{At}z)) = \frac{\partial}{\partial t}[e^{-At}P^k(e^{At}z)], \qquad k \in Z; \tag{30}$$

$$F^k_{k-1}(x) = F^k_{k-2}(x) + AP^k(x) - DP^k(x)Ax. \tag{31}$$

According to equations (30) and (31), one has

$$e^{-At}F^k_{k-1}(e^{At}z) + \frac{\partial}{\partial t}[e^{-At}P^k(e^{At}z)] = e^{-At}F^k_{k-2}(e^{At}z). \tag{32}$$

Suppose that

$$P^k(x) = \begin{bmatrix} \sum_{m+n=k} a_{mn}x_1^m x_2^n \\ \\ \sum_{m+n=k} b_{mn}x_1^m x_2^n \end{bmatrix};$$

then, using

$$x = e^{At}z = \begin{pmatrix} e^{\lambda_1 t}z_1 \\ e^{\lambda_2 t}z_2 \end{pmatrix},$$

one has

$$e^{-At}P^k(e^{At}z) = \begin{bmatrix} \sum_{m+n=k} a_{mn}\,e^{(-\lambda_1 + m\lambda_1 + n\lambda_2)t}z_1^m z_2^n \\ \\ \sum_{m+n=k} b_{mn}\,e^{(-\lambda_2 + m\lambda_1 + n\lambda_2)t}z_1^m z_2^n \end{bmatrix}.$$

Thus,

$$\frac{\partial}{\partial t}[e^{-At}P^k(e^{At}z)] = \begin{bmatrix} \sum_{\substack{m+n=k \\ -\lambda_1 + m\lambda_1 + n\lambda_2 \neq 0}} a_{mn}(-\lambda_1 + m\lambda_1 + n\lambda_2)\,e^{(-\lambda_1 + m\lambda_1 + n\lambda_2)t}z_1^m z_2^n \\ \\ \sum_{\substack{m+n=k \\ -\lambda_2 + m\lambda_1 + n\lambda_2 \neq 0}} b_{mn}(-\lambda_2 + m\lambda_1 + n\lambda_2)\,e^{(-\lambda_2 + m\lambda_1 + n\lambda_2)t}z_1^m z_2^n \end{bmatrix}, \tag{33}$$

According to equations (25), (32) and (33), one reaches the following conclusion:

$$\underset{t}{M}\{e^{-At}[F^k_{k-2}(e^{At}z) - F^k_{k-1}(e^{At}z)\} = \underset{t}{M}\left\{\frac{\partial}{\partial t}[e^{-At}P^k(e^{At}z)]\right\} = 0. \tag{34}$$

Thus, one has

$$\underset{t}{M}\{e^{-At}[F^k_{k-1}(e^{At}z)\} = \underset{t}{M}\{[e^{-At}F^k_{k-2}(e^{At}z)\}. \tag{35}$$

From equations (26) and (35), one has

$$F^k_{k-1}(z) = G^k(z) = \underset{t}{M}\{e^{-At}F^k_{k-1}(e^{At}z)\} = \underset{t}{M}\{e^{-At}F^k_{k-2}(e^{At}z)\}. \tag{36}$$

From equation (32), one has

$$P^k(z) = e^{-At}P^k(e^{At}z)|_{t=0} = \int \{e^{-At}[F^k_{k-2}(e^{At}z) - F^k_{k-1}(e^{At}z)]\}\,dt|_{t=0}$$

$$= \int \{e^{-At}F^k_{k-2}(e^{At}z) - G^k(z)\}\,dt|_{t=0}. \tag{37}$$

According to equations (36) and (37), $F^k_{k-1}(z)$ and $P^k(z)$ can be obtained directly from the $(k-2)$th transformed functions $F^k_{k-2}(z)$, where $k \geqslant 2$. From equation (37), $P^k(z)$ can be expressed as polynomials of order $k$ easily. Then $F^k_{k-1}(z)$ and $P^k(z)$ can be obtained conveniently as follows:

$$F^2(z) \Rightarrow \begin{cases} G^2(z) \\ P^2(z) \end{cases} \xrightarrow{z = y + P^2(y)} \begin{cases} F^k_1(y) \\ F^3_1(y) \end{cases} \Rightarrow \begin{cases} G^3(y) \\ P^3(y) \end{cases} \xrightarrow{y = x + P^3(x)} \begin{cases} F^k_2(x) \\ F^4_2(x) \end{cases} \cdots .$$

In the above calculations, one does not have to solve any equations; only simple iterations, like $z = y + P^k(y)$, are involved. This makes the calculations of normal forms very convenient. The above results can be obtained rather easily using symbolic computation. The convenience comes from the transformation $y = e^{At}z$. It seems that this transformation is usually employed in averaging methods, but actually there are some major differences between the methods. Averaging methods (such as those used in reference [9]), may lead to correct resonant polynomials, giving the right form, but this is of course not enough for a complete normal form; the coefficients must also be determined correctly.

Consider the function

$$F(x) = \varepsilon F^2(x) + \varepsilon^2 F^3(x) + \varepsilon^3 F^4(x) + \varepsilon^4 F^5(x) + \varepsilon^5 F^6(x), \tag{38}$$

the transformations

$$x = y + \varepsilon\phi^2(y) + \varepsilon^2\phi^3(y) + \varepsilon^3\phi^4(y) + \varepsilon^4\phi^5(y) + \cdots, \tag{39}$$

and

$$\begin{aligned} x &= u + \varepsilon\phi^2(u) \\ u &= v + \varepsilon^2\phi^3(v) \\ v &= w + \varepsilon^3\phi^4(w) \\ w &= y + \varepsilon^4\phi^5(y) \\ &\cdots. \end{aligned} \tag{40}$$

It is evident that the two sets of transformations (39) and (40) are not identical to each other. Actually, the averaging methods, such as the KB averaging method, the KBM averaging method, and the multiple time scales method [9] basically use transformations similar to equation (39), but the normal form theory uses transformations similar to equation (40). Before comparing all the coefficients of related terms (not just up to order 3), one cannot compare the results of two approaches properly. The results of averaging may be identical to those of normal form theory up to order 3 (as in reference [9]), but may not be so at higher orders. To see this, we note that substituting transformation (39) into function $F(x)$ results in

$$F_A(y) = \varepsilon^2 F_A^2 + \varepsilon^3 F_A^3 + \varepsilon^4 F_A^4 + \varepsilon^5 F_A^5 + \varepsilon^6 F_A^6. \tag{41}$$

Substituting transformation (40) into function $F(x)$ results in

$$F_N(y) = \varepsilon^2 F_N^2 + \varepsilon^3 F_N^3 + \varepsilon^4 F_N^4 + \varepsilon^5 F_N^5 + \varepsilon^6 F_N^6, \tag{42}$$

where $F_A^k$ and $F_N^k$ are the transformed functions with $\varepsilon$-order $k$.

Comparing the coefficients of same-ordered terms in equations (41) and (42) produces

$$F_A^2 - F_N^2 = 0, \qquad F_A^3 - F_N^3 = 0, \qquad F_A^4 - F_N^4 = DF^2DP^2P^3,$$
$$F_A^5 - F_N^5 = DF^3DP^2P^3 + DF^2DP^2P^4 + D^2F^2DP^2P^2P^3. \tag{43}$$

Clearly, the two approaches give identical results up to order 3, but the right sides of the last two equations have to be zero in order to have identical results up to order 5.

## 4. GENERAL SOLUTION

Consider the equation

$$\dot{x} = Bx + \sum_{s=2}^{k} E^s(x), \tag{44}$$

where $x \in R^2$, $E^s \in H_2^s$; $H_2^s$ is the bi-variate polynomial space of order $s$. They are defined by

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}, \qquad E^s(x) = \begin{bmatrix} \sum_{i+j=s} a_{ij} x_1^i x_2^j \\ \\ \sum_{i+j=s} b_{ij} x_1^i x_2^j \end{bmatrix},$$

$$i, j = 0, \ldots, s, \quad s \geqslant 2,$$

where $a_{ij}$ and $b_{ij}$ are constants and $\pm\omega$ are the eigenvalues.

First of all, transforming equation (44) into complex co-ordinates using

$$x_1 = \tfrac{1}{2}(z_1 + z_2), \qquad x_2 = \frac{1}{2i}(z_1 - z_2)$$

yields

$$\dot{z} = Az + \sum_{s=2}^{k} F^s(z), \tag{45}$$

where

$$A = \begin{pmatrix} \mathrm{i}\omega & 0 \\ 0 & -\mathrm{i}\omega \end{pmatrix}, \qquad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \qquad z_2 = \bar{z}_1.$$

Suppose that $z = x + P^s(x)$, $s = 2, \ldots, k$, $P^5 \in H_2^s$. Substituting these equations into equation (45) and truncating the $k$th term of the Taylor series leads to

$$\dot{x} = Ax + \sum_{i=2}^{k} F_n^i(x), \qquad n = 1, 2, \ldots, i-1, \tag{46}$$

where

$$\begin{aligned}
F_1^2 &= F^2 + AP^2 - DP^2Ax,\\
F_1^3 &= F^3 + DF^2P^2 - DP^2F_1^2,\\
F_1^4 &= F^4 + \tfrac{1}{2}D^2F^2(P^2)^2 + DF^3P^2 - DP^2F_1^3,\\
F_1^5 &= F^5 + \tfrac{1}{2}D^2F^3(P^2)^2 + DF^4P^2 - DP^2F_1^4,\\
F_2^3 &= F_1^3 + AP^3 - DP^3Ax,\\
F_2^4 &= F_1^4 + DF_1^2P^3 - DP^3F_1^2,\\
F_2^5 &= F_1^5 + DF_1^3P^3 - DP^3F_2^3,\\
F_3^4 &= F_2^4 + AP^4 - DP^4Ax,\\
F_3^5 &= F_2^5 + DF_1^2P^4 - DP^4F_1^2,\\
F_4^5 &= F_3^5 + AP^5 - DP^5Ax,\\
&\cdots.
\end{aligned}$$

$F_m^n(x)$ can easily be obtained by symbolic calculation.

Note that the $k$th equation is given by

$$\dot{x} = Ax + \sum_{i=2}^{k} F_{i-1}^i(x). \tag{47}$$

Suppose that $x = e^{At}y$. Substituting this equation into equation (47) results in

$$\dot{y} = \sum_{i=2}^{k} e^{-At}F_{i-1}^i(e^{At}y). \tag{48}$$

According to equation (28), above equation can be expressed as

$$\dot{y} = \sum_{s=2}^{k} e^{-At}F_{s-1}^s(e^{At}y) = \sum_{s=2}^{k} G^s(y). \tag{49}$$

According to equation (36), $G^s(y)$ can be obtained as

$$G^s(y) = \begin{Bmatrix} \displaystyle\sum_{\substack{m+n=s\\ m-n-1=0}} a_{mn}^{s-2} y_1^m y_2^n \\ \displaystyle\sum_{\substack{m+n=s\\ m-n+1=0}} b_{mn}^{s-2} y_1^m y_2^n \end{Bmatrix}, \qquad G^{2q} = 0,$$

in which $a_{ij}^{s-2}$ and $b_{ij}^{s-2}$ are the coefficients in functions $F_{s-2}^s$, $2q \leqslant k$.

According to equation (37), $P^k(y)$ can be obtained as follows:

$$P^2(y) = e^{-At}P^2(e^{At}y)|_{t=0} = \frac{1}{i\omega}\begin{pmatrix} a_{20}y_1^2 - a_{11}y_1y_2 - \frac{1}{3}a_{02}y_2^2 \\ \frac{1}{3}b_{20}y_1^2 + b_{11}y_1y_2 - b_{02}y_2^2 \end{pmatrix},$$

$$P^3(y) = e^{-At}P^3(e^{At}y)|_{t=0} = \frac{1}{2i\omega}\begin{pmatrix} a^1_{30}y_1^3 - a^1_{12}y_1y_2^2 - \frac{1}{2}a^1_{03}y_2^3 \\ \frac{1}{2}b^1_{30}y_1^3 + b^1_{21}y_1^2y_2 - b^1_{03}y_2^3 \end{pmatrix},$$

$$P^4(y) = e^{-At}P^4(e^{Aty})|_{t=0} = \frac{1}{i\omega}\begin{pmatrix} \frac{1}{3}a^2_{40}y_1^4 + a^2_{31}y_1^3y_2 - a^2_{22}y_1^2y_2^2 - \frac{1}{3}a^2_{13}y_1y_2^3 - \frac{1}{5}a^2_{04}y_2^4 \\ \frac{1}{5}b^2_{40}y_1^4 + \frac{1}{3}b^2_{31}y_1^3y_2 + b^2_{22}y_1^2y_2^2 - b^2_{13}y_1y_2^3 - \frac{1}{3}b^2_{04}y_2^4 \end{pmatrix},$$

$$P^5(y) = e^{-At}P^5(e^{Aty})|_{t=0} = \frac{1}{2i\omega}\begin{pmatrix} \frac{1}{2}a^3_{50}y_1^5 + a^3_{41}y_1^4y_2 - a^3_{23}y_1^2y_2^3 - \frac{1}{2}a^3_{14}y_1y_2^4 - \frac{1}{3}a^3_{05}y_2^5 \\ \frac{1}{3}b^3_{50}y_1^5 + \frac{1}{2}b^3_{41}y_1^4y_2 + b^3_{23}y_1^2y_2^3 - b^3_{14}y_1y_2^4 - \frac{1}{2}b^3_{05}y_2^5 \end{pmatrix},$$

$$\cdots\cdot$$

$$P^k(y) = e^{-At}P^k(e^{At}y)|_{t=0} = \int [e^{-At}F^k_{k-2}(e^{At}y) - G^k(y)]\,dt|_{t=0}$$

$$= \frac{1}{i\omega}\left.\begin{pmatrix} \displaystyle\sum_{\substack{s=0 \\ k-2s-1\neq 0}}^{k} \frac{1}{k-2s-1}\, e^{i(k-2s-1)\omega t}a^{k-2}_{(k-s)s}y_1^{k-s}y_2^s \\ \displaystyle\sum_{\substack{s=0 \\ k-2s+1\neq 0}}^{k} \frac{1}{k-2s+1}\, e^{i(k-2s+1)\omega t}b^{k-2}_{(k-s)s}y_1^{k-s}y_2^s \end{pmatrix}\right|_{t=0}$$

$$= \frac{1}{i\omega}\begin{pmatrix} \displaystyle\sum_{\substack{s=0 \\ k-2s-1\neq 0}}^{k} \frac{1}{k-2s-1}\, a^{k-2}_{(k-s)s}y_1^{k-s}y_2^s \\ \displaystyle\sum_{\substack{s=0 \\ k-2s+1\neq 0}}^{k} \frac{1}{k-2s+1}\, b^{k-2}_{(k-s)s}y_1^{k-s}y_2^s \end{pmatrix},$$

where $a^{k-2}_{(k-s)s}$ and $b^{k-2}_{(k-s)s}$ are coefficients in function $F^k_{k-2}$.

It is evident that there are no resonant monomials in function $P^k(y)$.

An inverse transformation $y = e^{-At}z$ is carried out and substituted into equation (49). One has the normal form of equation (45) as follows:

$$\dot{z} = Az + \sum_{i=1}^{m} G^{2i+1}(z). \tag{50}$$

Transforming equation (50) into polar co-ordinates using

$$z_1 = r\,e^{i\theta}, \qquad z_2 = r\,e^{-i\theta},$$

one has

$$\dot{r} = \sum_{i=1}^{m} a_1 r^{2i+1}, \qquad \dot{\theta} = \omega + \sum_{i=1}^{m} b_1 r^{2i}, \tag{51}$$

where $2m + 1 \leqslant k$.

One can see from the above analysis that the $k$th order normal form can be obtained from the $(k-2)$th order transformed functions. Thus, using the new approach, it is very convenient to employ symbolic calculation.

## 5. HIGH-DIMENSIONAL SYSTEMS

The above analysis is valid for high-dimensional systems as well. Consider the equation

$$\dot{y} = Ay + \sum_{s=1}^{M} F^s(y), \tag{52}$$

where $y \in C^n$, $F^k \in H_n^k$ and $H_n^k$ is a vector space of homogeneous polynomials of degree $k$ and is described by $n$ variables. $A$ is an $n \times n$ matrix, $y = (y_1 \;\; y_2 \dots y_n)^{\mathrm{T}}$, $F^k = (F_{(1)}^k \;\; F_{(2)}^k \dots F_{(n)}^k)^{\mathrm{T}}$ and

$$F_{(m)}^2 = \sum_{s_1 + s_2 + \cdots + s_n = 2} a_{s_1 s_2 \dots s_n (m)} y_1^{s_1} y_2^{s_2} \dots y_n^{s_n},$$

$$F_{(m)}^3 = \sum_{s_1 + s_2 + \cdots + s_n = 3} a_{s_1 s_2 \dots s_n (m)} y_1^{s_1} y_2^{s_2} \dots y_n^{s_n},$$

$$\dots$$

$$F_{(m)}^k = \sum_{s_1 + s_2 + \cdots + s_n = k} a_{s_1 s_2 \dots s_n (m)} y_1^{s_1} y_2^{s_2} \dots y_n^{s_n}.$$

Suppose that $A = \mathrm{diag}\,(\lambda_1, \lambda_2, \dots, \lambda_n)$, where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues; all eigenvalues are pure imaginary pairs, where $2r = n$, $\lambda_{i+r} = \overline{\lambda}_i$, $i = 1, 2, \dots, r$.

Similarly, consider a series of transformations, which are similar to equations (15) and (17) in $C^n$, given by

$$z = y + P^k(y), \qquad k = 2, 3, \dots, M, \quad P^k \in H_n^k,$$

where $P^k(y)$ are undefined functions, which will be determined such that the terms of order $k$ in the transformed form will be simplified as resonant polynomial of order $k$.

Substituting the above transformations into equation (52) results in

$$\dot{y} = Ay + \sum_{s=2}^{M} F_{s-1}^s(y). \tag{53}$$

Suppose that $F_{k-1}^k(y) = G^k(y)$ in equation (53), where $G^k(y)$ are the resonant polynomials of order $k$. Introducing the transformation $y = \mathrm{e}^{At} z$ into equation (53), one has

$$\dot{z} = \mathrm{e}^{-At} \sum_{s=2}^{M} F_{s-1}^s(\mathrm{e}^{At} z). \tag{54}$$

Similarly, according to relation (32), one has

$$G^k(z) = \underset{t}{M}\{e^{-At}F^k_{k-2}(e^{At}z)\}, \qquad P^k(z) = \int \{e^{-At}F^k_{k-2}(e^{At}z) - Z^k(z)\}\, dt|_{t=0}.$$

Solving the above equation for $P^k(z)$, the coefficients in $P^k(z)$ and $G^k(z)$ can be obtained as follows:

$$G^k(z) = \begin{Bmatrix} \sum\limits_{\substack{\bar{s}=k \\ \delta-\lambda_1=0}} a^{k-2}_{s_1 s_2 \ldots s_n(1)} z_1^{s_1} z_2^{s_2} \ldots z_n^{s_n} \\ \sum\limits_{\substack{\bar{s}=k \\ \delta-\lambda_2=0}} a^{k-2}_{s_1 s_2 \ldots s_n(2)} z_1^{s_1} z_2^{s_2} \ldots z_n^{s_n} \\ \ldots \\ \sum\limits_{\substack{\bar{s}=k \\ \delta-\lambda_n=0}} a^{k-2}_{s_1 s_2 \ldots s_n(n)} z_1^{s_1} z_2^{s_2} \ldots z_n^{s_n} \end{Bmatrix}, \tag{55}$$

where $\bar{s} = s_1 + s_2 + \cdots + s_n$, $\delta = s_1\lambda_1 + s_2\lambda_2 + \cdots + s_n\lambda_n$, $k \geqslant 2$, $a^{k-2}_{s_1 s_2 \ldots s_n(m)}$ are the coefficients of transformed function $F^k_{k-2}(z)$; and

$$P^k(z) = \begin{Bmatrix} \sum\limits_{\substack{\bar{s}=k \\ \delta-\lambda_2\neq 0}} \dfrac{1}{\delta-\lambda_1} a^{k-2}_{s_1 s_2 \ldots s_n(1)} z_1^{s_1} z_2^{s_2} \ldots z_n^{s_n} \\ \sum\limits_{\substack{\bar{s}=k \\ \delta-\lambda_2\neq 0}} \dfrac{1}{\delta-\lambda_2} a^{k-2}_{s_1 s_2 \ldots s_n(2)} z_1^{s_1} z_2^{s_2} \ldots z_n^{s_n} \\ \ldots \\ \sum\limits_{\substack{\bar{s}=k \\ \delta-\lambda_n\neq 0}} \dfrac{1}{\delta-\lambda_n} a^{k-2}_{s_1 s_2 \ldots s_n(1)} z_1^{s_1} z_2^{s_2} \ldots z_n^{s_n} \end{Bmatrix}, \tag{56}$$

Introducing the inverse transformation $z = e^{-At}y$ into equation (54), one has

$$\dot{y} = Ay + \sum_{s=2}^{M} G^s(y). \tag{57}$$

This is the normal form of equation (52).

## 6. EXAMPLES

*Example* 1. Determine the normal form of the following two-dimensional system:

$$\dot{x} = Bx + E^2(x) + E^3(x), \tag{58}$$

where

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad E^2(x) = \begin{pmatrix} a_{20}x_1^2 + a_{11}x_1x_2 + a_{02}x_2^2 \\ b_{20}x_1^2 + b_{11}x_1x_2 + b_{02}x_2^2 \end{pmatrix},$$

$$E^3(x) = \begin{pmatrix} a_{30}x_1^3 + a_{21}x_1^2x_2 + a_{12}x_1^1x_2^2 + a_{03}x_2^3 \\ b_{30}x_1^3 + b_{21}x_1^2x_2 + b_{12}x_1^1x_2^2 + b_{03}x_2^3 \end{pmatrix}.$$

First of all, transforming equation (58) into the complex form by using

$$x_1 = \tfrac{1}{2}(z_1 + z_2), \qquad x_2 = \frac{1}{2i}(z_1 - z_2)$$

yields

$$\dot{z} = Az + F^2(z) + F^3(z), \tag{59}$$

where

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \qquad z_2 = \bar{z}_1.$$

Suppose that $z = u + P^2(u)$, $P^2 \in H_2^2$. Substituting this equation into equation (59) leads to

$$\dot{u} = Au + F_1^2(u) + F_1^3(u).$$

Suppose that $u = x + P^3(x)$, $P^3 \in H_2^3$. Substituting this equation into the above equation leads to

$$\dot{x} = Ax + F_1^2(x) + F_2^3(x).$$

Suppose that $F_{k-1}^k(y) = G^k(y)$, where $k = 1, 2$; $G^k(y)$ are the resonant polynomials of order $k$. Introducing the transformation $x = e^{At}y$ and substituting it into above equation procedures

$$\dot{y} = e^{-At}F_1^2(e^{At}y) + e^{-At}F_1^3(e^{At}y).$$

Suppose

$$F_{k-1}^k(y) = \begin{pmatrix} F_{k-1(1)}^k(y) \\ F_{k-1(2)}^k(y) \end{pmatrix} = \begin{bmatrix} \sum_{m+n=k} a_{mn(1)}^{k-1} y_1^m y_2^n \\ \\ \sum_{m+n=k} a_{mn(2)}^{k-1} y_1^m y_2^n \end{bmatrix},$$

where $k = 1, 2$. According to equation (26), one has

$$e^{-At}F_1^2(e^{At}y) + e^{-At}F_1^3(e^{At}y) = F_1^2(y) + F_2^3(y).$$

Then, one has

$$\dot{y} = \underset{t}{M}\{e^{-At}F_1^2(e^{At}y) + e^{-At}F_1^3(e^{At}y)\} = G^2(y) + G^3(y).$$

Introducing the inverse transformation $y = e^{-At}x$ in the above equation, according to equation (27), one has

$$\dot{x} = Ax + G^2(x) + G^3(x) + \text{h.o.t.}$$

This is the normal form of equation (59). Transforming to polar co-ordinates, one has

$$\dot{r} = a_1 r^3, \qquad \dot{\theta} = \omega + b_1 r^2.$$

According to equations (36) and (37), $P^2$ and $G^2$ can be obtained from function $F^2$; $F_1^3$ can be obtained from $P^2$ and $F^3$; $G^3$ can be obtained from $F_1^3$. Then, $G^2 = M_t\{e^{-At}F^2(e^{At}y)\}$, $G^3 = M_t\{e^{-At}F_1^3(e^{At}y)\}$. The coefficients $a_1$ and $b_1$ can be obtained as follows using MAPLE. The calculation time on a PC (CPU 200) computer is very short (0·1 s).

$$a_1 = \tfrac{1}{8}(b_{21} + 3a_{30} + 3b_{03} + a_{12} - b_{02}b_{11} + 2a_{02}b_{02} + a_{11}a_{02} + a_{11}a_{20} - b_{20}b_{11} - 2b_{20}a_{20}),$$

$$b_1 = \tfrac{1}{24}(-3a_{21} + 9b_{30} - 9a_{03} + 3b_{12} - 4b_{02}^2 + 5b_{02}a_{11} - 10b_{02}b_{20} - a_{11}^2 + a_{11}b_{20} - 10b_{20}^2$$
$$- 10a_{02}^2 + a_{02}b_{11} - 10a_{02}a_{20} - b_{11}^2 + 5b_{11}a_{20} - 4a_{20}^2), \tag{60}$$

where $a_{ij}$ and $b_{ij}$ are the coefficients of functions $E^2$ and $E^3$.

*Example* 2. Determine the normal forms and related coefficients of the two-dimensional system with six parameters, given by

$$\dot{x} = -y + \lambda_1 x - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2,$$
$$\dot{y} = x + \lambda_1 y + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2. \tag{61}$$

This is a popular model to study Hilbert's 16th problem. It was studied originally by Bautin [14] in 1954. However, he obtained a wrong result by using the method of successive functions. Thirty years later, Farr *et al*. [15] studied this problem again using the Lyapunov–Schmidt method, and corrected Bautin's mistake in 1989. Now we use the proposed new approach to study this problem. Transforming the above equation into complex co-ordinate form, one has

$$\dot{z} = Az + F^2(z), \tag{62}$$

where

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad F^2(x) = \begin{pmatrix} a_{20}z_1^2 + a_{11}z_1z_2 + a_{02}z_2^2 \\ b_{20}z_1^2 + b_{11}z_1z_2 + b_{02}z_2^2 \end{pmatrix}, \qquad a_{mn} = a_{mn}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6).$$

Following the same procedures as introduced above, introduce a set of transformations $z = y + P^k(y)$, $k = 2, \ldots, 7$ into equation (62) to obtain

$$\dot{y} = iy + F_1^2(y) + F_2^3(y) + F_3^4(y) + F_4^5(y) + F_5^6(y) + F_6^7(y), \tag{63}$$

where $F_{k-1}^k(y)$ can be calculated similar to equation (46).

It is easy to obtain the normal form in polar co-ordinates as follows:

$$\dot{r} = a_1 r^3 + a_2 r^5 + a_3 r^7, \qquad \dot{\theta} = 1 + O(|r|^2). \tag{64}$$

According to the analysis in section 3, after introducing the transformation $y = e^{At}x$ in equation (63), the calculation of coefficients of normal forms can be simplified as

$$F_2^3 = M_t\{e^{-At}F_1^3(e^{At}x)\}, \qquad F_4^5 = M_t\{e^{-At}F_3^5(e^{At}x)\} = M_t\{e^{-At}F_2^5(e^{At}x)\},$$

$$F_6^7 = M_t\{e^{-Art}F_5^7(e^{At}x)\} = M_t\{e^{-Art}F_3^7(e^{At}x)\}, \qquad F_1^2 = F_3^4 = F_5^6 = 0.$$

Then the coefficients of the normal form in equation (64) can be obtained as follows:

$$a_1 = -\tfrac{1}{8}\lambda_5(\lambda_3 - \lambda_6), \qquad \text{for } \lambda_1 = 0,$$

$$a_2 = \tfrac{1}{48}\lambda_2\lambda_4(\lambda_3 - \lambda_6)[\lambda_4 + 5(\lambda_3 - \lambda_6)], \qquad \text{for } \lambda_1 = \lambda_5 = 0,$$

$$a_3 = \tfrac{25}{64}\lambda_2(\lambda_3 - \lambda_6)^3(\lambda_3\lambda_6 - \lambda_2^2 - 2\lambda_6^2), \qquad \text{for } \lambda_1 = \lambda_5 = 0, \quad \lambda_4 = -5(\lambda_3 - \lambda_6). \quad (65)$$

These are identical to the results of Farr *et al*. [15], who employed L–S theory, and the same as the conclusion Yu and Huseyin [16], who used the intrinsic harmonic balancing technique and multiple time scales method. As stated in reference [15], using the L–S method to solve this example requires "a long calculation". The normal form method introduced here gives the results readily and conveniently. Indeed, using MAPLE, the coefficients in equation (64) are obtained within 2 s on a PC (CPU 200) computer (to be exact in 1·4 s).

*Example* 3. Determine the normal forms and related coefficients of the following Duffing equation:

$$\dot{x} = y, \qquad \dot{y} = -x + \alpha x^3. \quad (66)$$

First of all, transforming equation (66) into complex form by using

$$x = \tfrac{1}{2}(z + \bar{z}), \qquad y = \frac{1}{2i}(z - \bar{z})$$

yields

$$\dot{z} = -\mathrm{i}z + \frac{\mathrm{i}\alpha}{8}(z + \bar{z})^3. \quad (67)$$

Following the same procedures as in Example 1, introduce a set of transformations $z = y + P^k(y)$, $k = 2, \ldots, 5$, into equation (67), to obtain

$$\dot{y} = -\mathrm{i}y + F_1^2(y) + F_2^3(y) + F_3^4(y) + F_4^5(y), \quad (68)$$

where $P^k(y)$ are undefined functions, which will be determined such that the terms of order $k$ in the transformed form will be simplified as a resonant polynomial of order $k$; $F_{k-1}^k(y)$ are transformed functions that have been calculated in equation (46). According to the above analysis, $P^k(y)$ can be solved from $F_{k-1}^k(y) = G^k(y)$. Transforming equation (68) to polar co-ordinates, one has

$$\dot{r} = a_1 r^3 + a_2 r^5, \qquad \dot{\theta} = \omega + b_1 r^2 + b_2 r^4. \quad (69)$$

In this example, $F^2 = 0$. According to the analysis in section 3, after introducing the transformation $y = \mathrm{e}^{At}x$ into equation (68), the calculation of coefficients of normal form can be simplified as

$$F_2^3 = \underset{t}{M}\{\mathrm{e}^{-Art}F_1^3(\mathrm{e}^{At}x)\} = \underset{t}{M}\{\mathrm{e}^{-Art}F^3(\mathrm{e}^{At}x)\},$$

$$F_4^5 = \underset{t}{M}\{\mathrm{e}^{-At}F_3^5(\mathrm{e}^{At}x)\} = \underset{t}{M}\{\mathrm{e}^{-At}F_2^5(\mathrm{e}^{At}x)\}.$$

Following the same procedures as introduced above, the coefficients of the above equation are obtained as follows:

$$a_1 = \frac{3\alpha}{8}, \quad b_1 = 0; \qquad a_2 = 0, \quad b_2 = \frac{27\alpha^2}{256}.$$

Using MAPLE, the coefficients in equation (69) are obtained within 1 s on a PC (CPU 200) computer (to be exact in 0·1 s).

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